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Calculation of the anomalous $\gamma\pi^* \rightarrow \pi\pi$ form factor

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Abstract

The form factor for the anomalous process $\gamma\pi^* \rightarrow \pi\pi$, $F^{3\pi}(s, t, u)$, is calculated as a phenomenological application of the QCD Dyson-Schwinger equations. The chiral-limit value dictated by the electromagnetic, anomalous chiral Ward identity, $F^{3\pi}(0, 0, 0) = eN_c/(12\pi^2 f_\pi^3)$, is reproduced, *independent* of the details of the modelling of the gluon and quark 2-point Schwinger functions. Using a parametrisation of the dressed u - d quark 2-point Schwinger function that provides a good description of pion observables, such as π - π scattering-lengths and the electromagnetic pion form factor, $F^{3\pi}(s, t, u)$ is calculated on a kinematic range that proposed experiments plan to explore. Our result confirms the general trend of other calculations; i.e., a monotonic increase with s at fixed t and u , but is uniformly larger and exhibits a more rapid rise with s .

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1. Introduction. Hadronic processes involving an odd number of pseudoscalar mesons are of particular interest because they are intimately connected to the anomaly structure of QCD. The decay $\pi^0 \rightarrow \gamma\gamma$ is the primary example of such an anomalous process. That such processes occur in the chiral limit ($m_\pi^2 = 0$) is a fundamental consequence of the quantisation of QCD; i.e., of the non-invariance of the QCD measure under chiral transformations even in the absence of current quark masses [1]. The $\pi^0 \rightarrow \gamma\gamma$ decay rate can be calculated from a quark triangle diagram and agreement with the observed rate requires that the number of colours, N_c , equal three. The transition form factor for the related process $\gamma^*\pi^0 \rightarrow \gamma$ can be measured experimentally [2] and has attracted keen theoretical interest [3,4] because it involves only one hadronic bound state and provides a good test of QCD-based models and their interpolation between the soft and hard domains.

Another anomalous form factor, accessible to experiment, is that which describes the transition $\gamma\pi^* \rightarrow \pi\pi$, denoted by $F^{3\pi}(s, t, u)$. This provides additional constraints on QCD based-models because three hadronic bound states are involved. The form factor $F^{3\pi}(s, t, u)$ has been measured at Serpukhov in the Primakov reaction $\pi^- A \rightarrow \pi'^-\pi^0 A'$. [5] In this experiment the considerable uncertainty in both the kinematic range and result make it difficult to draw a conclusion regarding the accuracy of the theoretical prediction for the chiral limit value of $F^{3\pi}(0, 0, 0)$ [6]. New experiments are planned at CEBAF: [7] $\gamma p \rightarrow \pi^+\pi^0 n$, $s/m_\pi^2 \in [4, 15]$; and at FermiLab [8] via the Primakov reaction using a 600 GeV pion beam, $s/m_\pi^2 \in [4, 6]$.

Herein we report a calculation of $F^{3\pi}(s, t, u)$ on the kinematic range to be explored in the CEBAF experiment [7]. We evaluate the amplitude indicated by the diagram in Fig. 1, in which the quark 2-point Schwinger function, quark-photon vertex and quark-pion vertex (pion Bethe-Salpeter amplitude) are dressed quantities whose form follows from the extensive body of nonperturbative, semi-phenomenological Dyson-Schwinger equations studies in QCD. [9,10] In this way our calculation provides for an extrapolation of the known large spacelike- q^2 behaviour of these QCD Schwinger functions to the small spacelike- q^2 region, where they are unknown and confinement effects are manifest. This facilitates an exploration of the relationship of physical observables to the nonperturbative, infrared behaviour of these Schwinger functions.

This calculation employs the model forms introduced in Ref. [10]. The quark 2-point Schwinger function has no Lehmann representation and hence may be interpreted as describing a confined particle since this feature is sufficient to ensure the absence of quark production thresholds in S -matrix elements describing colour-singlet to singlet transitions. The quark-photon vertex, which describes the coupling of a photon to a dressed quark, follows from extensive QED studies [11,12] and satisfies the Ward-Takahashi identity. This necessarily entails that the amplitude is current conserving. In the chiral limit the quark-pion vertex is completely determined by the quark 2-point Schwinger function [13], which is the manifestation of Goldstone's theorem in this approach. The extension to finite quark mass requires a minimal modification and preserves Dashen's relation [14].

The quark 2-point Schwinger function is parametrised such that it provides an good description of π - π scattering at, and to approximately 600 MeV above, threshold and the electromagnetic pion form factor at spacelike momentum transfer. No adjustment of the parameters is made in calculating $F^{3\pi}(s, t, u)$. The $\gamma\pi^* \rightarrow \pi\pi$ process probes the pion bound state amplitude well outside the domain of the complex plane on which it has been fitted;

i.e., well into the timelike region, and hence provides a new test of the model. Therefore, interpreted within our framework, the experimental determination of $F^{3\pi}(s, t, u)$ provides important new information about the structure of the nonperturbative pion bound state amplitude.

2. The amplitude for $\gamma\pi^* \rightarrow \pi\pi$. In Euclidean space, with metric $\delta_{\mu\nu} = \text{diag}(1, 1, 1, 1)$ and $\gamma_\mu = \gamma_\mu^\dagger$, the amplitude for the $\gamma\pi^*\pi\pi$ vertex (see Fig. 1) is

$$\begin{aligned} & i\epsilon_{\mu\nu\rho\sigma} p_{1\nu} p_{2\rho} p_{3\sigma} F^{3\pi}(p_1, p_2, p_3) = \\ & 2eN_c \int \frac{d^4k}{(2\pi)^4} \text{tr}_D [\Gamma_\mu(k_{---}, k_{+++}) S(k_{+++}) \gamma_5 \Gamma_\pi(k_{0++}) S(k_{-++}) \\ & \gamma_5 \Gamma_\pi(k_{-0+}) S(k_{--+}) \gamma_5 \Gamma_\pi(k_{--0}) S(k_{---})] . \end{aligned} \quad (1)$$

In this expression the colour and flavour traces have been evaluated, leaving only the Dirac trace, and we have employed the definition $k_{\alpha\beta\gamma} = k + \frac{\alpha}{2}p_1 + \frac{\beta}{2}p_2 + \frac{\gamma}{2}p_3$. Note that due to momentum conservation the photon momentum is equal to the sum of pion momenta: $q = p_1 + p_2 + p_3$. In the following we adopt the convention that the pions labelled 1 and 2 are on-shell whereas pion 3 is off-shell. The dressed quark-photon vertex is denoted by $\Gamma_\mu(k_1, k_2)$, the pion Bethe-Salpeter amplitude by $\gamma_5 \Gamma_\pi(k)$ and the dressed quark 2-point Schwinger function by $S(k)$. Equation (1) can be derived as an application of the formalism described in Ref. [15]. Evaluated with a dressed quark-photon vertex that satisfies the Ward-Takahashi identity, which is a minimal requirement, this expression is current conserving.

The quark 2-point Schwinger function can be written

$$S(p) = -i\gamma \cdot p \sigma_V(p^2) + \sigma_S(p^2) \quad (2)$$

$$= \frac{1}{i\gamma \cdot p A(p^2) + m + B(p^2)} , \quad (3)$$

where $m = m_u = m_d$ is the current quark mass, and can be obtained by solving the quark Dyson-Schwinger equation (DSE) [9]. The many studies of this equation ensure that the qualitative features of the functions σ_S and σ_V are well known for real, spacelike- p^2 .

In Ref. [10], in order to avoid the need for a numerical solution of the quark DSE, the following *approximating* algebraic forms are used:

$$\bar{\sigma}_S(x) = C_m e^{-2x} + \frac{1 - e^{-b_1 x}}{b_1 x} \frac{1 - e^{-b_3 x}}{b_3 x} \left(b_0 + b_2 \frac{1 - e^{-\Lambda x}}{\Lambda x} \right) + \frac{\bar{m}}{x + \bar{m}^2} (1 - e^{-2(x + \bar{m}^2)}) \quad (4)$$

$$\bar{\sigma}_V(x) = \frac{2(x + \bar{m}^2) - 1 + e^{-2(x + \bar{m}^2)}}{2(x + \bar{m}^2)^2} - \bar{m} C_m e^{-2x}. \quad (5)$$

with $k^2 = 2Dx$, $\bar{\sigma}_S(x) = \sqrt{2D} \sigma_S(k^2)$, $\bar{\sigma}_V(x) = 2D \sigma_V(k^2)$ and $\bar{m} = m/\sqrt{2D}$. The quantity $\lambda = \sqrt{2D}$ is a mass-scale related to the infrared behavior of the gluon 2-point Schwinger function [9].

At large spacelike- p^2 the model forms behave as

$$\sigma_S(p^2) \approx \frac{m}{p^2} - \frac{m^3}{p^4} + \frac{b_0}{b_1 b_3} \frac{\lambda^3}{p^4} + \dots \quad \text{and} \quad \sigma_V(p^2) \approx \frac{1}{p^2} - \frac{D + m^2}{p^4} + \dots , \quad (6)$$

whereas in QCD one has

$$\sigma_S(p^2) \approx \frac{\hat{m}}{p^2 \left[\frac{1}{2} \ln \left(p^2 / \Lambda_{\text{QCD}}^2 \right) \right]^d} - \frac{4\pi^2 d}{3} \frac{\langle \bar{q}q \rangle}{p^4 \left[\ln \left(p^2 / \Lambda_{\text{QCD}}^2 \right) \right]^{d-1}} + \dots \quad (7)$$

with \hat{m} and $\langle \bar{q}q \rangle$ the renormalisation point invariant current mass and condensate, respectively, and $d = 12/[33 - 2N_f]$; N_f is the number of quark flavours. Comparing these two equations one sees that, setting $d = 1$ and neglecting $\ln[p^2]$ terms, the model defined by Eqs. (4) and (5) properly represents the ultraviolet behaviour of the quark 2-point Schwinger function; incorporating asymptotic freedom and dynamical chiral symmetry breaking, with

$$-\langle \bar{q}q \rangle = \frac{3}{4\pi^2} \frac{b_0}{b_1 b_3} \lambda^3. \quad (8)$$

Another feature of this model is that both σ_S and σ_V are entire functions in the complex- p^2 plane but for an essential singularity. As a consequence the quark 2-point Schwinger function does not have a Lehmann representation and can be interpreted as describing a confined particle. This is because, when used in Eq. (1), for example, this property ensures the absence of free-quark production thresholds, under the reasonable assumptions that $\Gamma_\pi(p^2)$ is regular in the domain of integration. It follows from this that Eq. (1) is free of endpoint and pinch singularities.

The expressions in Eqs. (4) and (5) provide a six-parameter model of the dressed quark 2-point Schwinger function in QCD: $C_m, \bar{m}, b_0, \dots, b_3$. ($\Lambda = 10^{-4}$ is introduced simply to decouple b_2 from the quark condensate.) These parameters can easily be fitted to experimental observables, as we discuss below.

The Bethe–Salpeter amplitude, Γ_π in Eq. (1), is the solution of the homogeneous Bethe–Salpeter equation (BSE). Many studies of this BSE suggest strongly that the amplitude is dominantly pseudoscalar (as already indicated by the notation used in Eq. (1)). Furthermore, in the chiral limit the pseudoscalar BSE and quark DSE are identical [13] and one has a massless excitation in the pseudoscalar channel with

$$\Gamma_\pi(p; P^2 = 0) = \frac{1}{f_\pi} B_{m=0}(p^2), \quad (9)$$

where $B_{m=0}(p^2)$ is given in Eq. (3) with $m = 0$. This is the realisation of Goldstone’s theorem in the DSE framework; i.e., in the chiral limit Eqs. (4) and (5) completely determine Γ_π .

Herein we employ the approximation

$$\Gamma_\pi(p; P^2 = -m_\pi^2) \approx \frac{1}{f_\pi} B_{m=0}(p^2), \quad (10)$$

which, for small current quark masses, is a good approximation both pointwise and in terms of the values obtained for physical observables. [16]

The quark-photon vertex, $\Gamma_\mu(p_1, p_2)$, satisfies a DSE that describes both strong and electromagnetic dressing of the vertex. Solving this equation is a difficult problem that has only recently begun to be addressed [17]. However, much progress has been made in constraining the form of $\Gamma_\mu(p_1, p_2)$ and developing a realistic Ansatz [11,12]. It is obvious that the bare vertex, $\Gamma_\mu(p_1, p_2) = \gamma_\mu$, is inadequate when the fermion 2-point Schwinger function has momentum dependent dressing because it violates the Ward-Takahashi identity. In Ref. [18] the following form was proposed

$$\Gamma_\mu^{\text{BC}}(p, k) = \frac{[A(p^2) + A(k^2)]}{2} \gamma_\mu + \frac{(p+k)_\mu}{p^2 - k^2} \left\{ [A(p^2) - A(k^2)] \frac{[\gamma \cdot p + \gamma \cdot k]}{2} - i [B(p^2) - B(k^2)] \right\}. \quad (11)$$

This Ansatz is *completely determined* by the dressed quark 2-point Schwinger function and has the features that it: 1) satisfies the Ward-Takahashi identity, thereby ensuring current conservation at the microscopic level; 2) is free of kinematic singularities - i.e., it has a well defined limit as $p^2 \rightarrow k^2$; 3) transforms correctly under C , P , T and Lorentz transformations; and 4) reduces to the bare vertex in the manner prescribed by perturbation theory. The fact that it is nevertheless relatively simple makes it an ideal form to be employed in phenomenological studies. The studies of Ref. [10] illustrate its phenomenological efficacy. In these studies, for example, it is shown that the chiral limit for the anomalous decay $\pi^0 \rightarrow \gamma\gamma$ is reproduced exactly using

$$\Gamma_\mu(p, k) = \Gamma_\mu^{\text{BC}}(p, k). \quad (12)$$

3. Chiral Limit: $\gamma\pi^* \rightarrow \pi\pi$. At the soft point in the chiral limit ($s = t = u = 0$) the transition form factor of Eq. (1) is

$$F^{3\pi}(0, 0, 0) = \frac{eN_c}{2\pi^2} \int_0^\infty ds s \Gamma_\pi(s)^3 \sigma_V \left\{ A \sigma_V \sigma_S + \frac{3}{2} s A \sigma'_V \sigma_S - \frac{3}{2} s A \sigma_V \sigma'_S + \frac{1}{2} s A' \sigma_V \sigma_S - \frac{1}{2} s B' \sigma_V^2 \right\}. \quad (13)$$

Defining $C(s) = B(s)^2/[s A(s)^2] = \sigma_S(s)^2/[s \sigma_V(s)^2]$ one obtains a dramatic simplification and Eq. (13) becomes

$$F^{3\pi}(0, 0, 0) = -\frac{eN_c}{2\pi^2} \int_0^\infty ds \frac{\Gamma_\pi(s)^3}{B(s)^3} \frac{C'(s)C(s)}{[1 + C(s)]^4}. \quad (14)$$

Recalling Eq. (9), which is the manifestation of Goldstone's theorem in the DSE approach, it follows that

$$F^{3\pi}(0, 0, 0) = \frac{eN_c}{2\pi^2 f_\pi^3} \int_0^\infty dC \frac{C}{(1 + C)^4} = \frac{eN_c}{12\pi^2 f_\pi^3}, \quad (15)$$

since $C(s)$ is a monotonic function for $s \geq 0$ with $C(s = 0) = \infty$ and $C(s = \infty) = 0$. Hence, the chiral limit value [6] is reproduced *independent* of the details of the quark 2-point Schwinger function, $S(p)$.

In order to obtain the result in Eq. (15) it is essential that, in addition to Eq. (9), the photon-quark vertex satisfy the Ward identity. This is not surprising. However, the fact that one must dress all of the elements in the calculation consistently is often overlooked. The subtle cancellations that are required to obtain this result also make it clear that it cannot be obtained in model calculations where an arbitrary cutoff function (or “form-factor”) is introduced into each integral. The fact that the pion Bethe-Salpeter amplitude is proportional to the scalar part of the quark self energy in the chiral limit, Eq. (9), is crucial. These features are also to be seen in the calculation [19] of the Wess-Zumino five pseudoscalar term and the $\pi^0\gamma\gamma$ vertex [10], in which again the values expected from considerations of

anomalous current conservation are obtained independent of the details of the quark 2-point Schwinger function, $S(p)$.

4. Numerical Results. The chiral limit value of the $\gamma\pi\pi\pi$ amplitude, $F^{3\pi}(0,0,0)$, provides a useful normalisation. We therefore define the function

$$\tilde{F}^{3\pi}(s, t, u) = F^{3\pi}(p_1, p_2, p_3)/F^{3\pi}(0, 0, 0) = \frac{1}{e} 4\pi^2 f_\pi^3 F^{3\pi}(p_1, p_2, p_3) . \quad (16)$$

We employ the following definition of the Mandelstam variables:

$$s = -(p_1 + p_2)^2 \equiv m_\pi^2 \bar{s}, \quad t = -p_3^2 \equiv m_\pi^2 \bar{t}, \quad u = -(p_1 + p_3)^2 \equiv m_\pi^2 \bar{u}, \quad (17)$$

which ensures that even though we work in Euclidean metric these variables have their conventional interpretation. We note that the quantity $t - m_\pi^2$ provides a measure of the amount by which the (third) pion is off-shell.

In the experiment proposed at CEBAF the photon energy in the proton rest frame is between 1 and 2 GeV. This suggests the following range of Mandelstam variables: $4 \leq \bar{s} \leq 16$, $-9 \leq \bar{t} \leq -1$, $-16 \leq \bar{u} \leq 5$. For the purpose of the numerical calculation we fix $\bar{t} = -1$; i.e., we choose pion 3 to be as close as (experimentally) possible to its mass shell, $\bar{t} = 1$. Within the range of s considered, the requirement of a fixed photon energy of 1 GeV in the proton rest frame entails

$$\bar{u} = (-1.5 - 4.28x + 0.012x^2) \quad \text{with} \quad x = (\bar{s}/4 - 1). \quad (18)$$

We have demonstrated above that in the chiral limit $F^{3\pi}(0,0,0) = eN_C/(12\pi^2 f_\pi^3)$ independent of the model parameters. The evolution of $F^{3\pi}(s, t, u)$ with s does depend on the model parameters, even in the chiral limit. As we have described above, Eqs. (4), (5), (9) and (12) provide a six-parameter model $[C_m, \bar{m}, b_0 \dots b_3]$ of the nonperturbative, dressed-quark substructure of the pion based on DSE studies. These parameters are fixed by requiring that the model reproduce, as well as possible, the following pion observables $f_\pi/\langle\bar{q}q\rangle_{1\text{GeV}^2}^{1/3} = 0.42 \pm 0.02$, $f_\pi r_\pi = 0.31 \pm 0.01$, $m_\pi^2/\langle\bar{q}q\rangle_{1\text{GeV}^2}^{2/3} = 0.40 \pm 0.03$; the dimensionless π - π scattering lengths (discussed in Refs. [20,21,22] with current experimental values presented in Table I) and the pion electromagnetic form factor on spacelike- $q^2 \in [0, 4] \text{ GeV}^2$ [10].

The values of observables are given by simple integral expressions involving the quark 2-point Schwinger function and pion Bethe-Salpeter amplitude; for example, [16]

$$f_\pi^2 = \frac{N_c}{8\pi^2} \int dp^2 p^2 B_0(p^2)^2 \left(\sigma_V^2 - 2[\sigma_S \sigma'_S + s \sigma_V \sigma'_V] - s[\sigma_S \sigma''_S - (\sigma'_S)^2] - s^2[\sigma_V \sigma''_V - (\sigma'_V)^2] \right) ; \quad (19)$$

$$m_\pi^2 f_\pi^2 = \frac{N_c}{2\pi^2} \int dp^2 p^2 \frac{B_0(p^2)}{B(p^2)} \left(B(p^2) \sigma_S^0(s) - B_0(p^2) \sigma_S(p^2) \right) , \quad (20)$$

which follows from the pion Bethe-Salpeter equation and is consistent with Dashen's relation [14]; and $\langle\bar{q}q\rangle_{\mu^2} = \ln[\mu^2/\Lambda_{\text{QCD}}^2] \langle\bar{q}q\rangle$ with $\langle\bar{q}q\rangle$ given in Eq. (8) and $\Lambda_{\text{QCD}} = 0.2 \text{ GeV}$. In these equations the subscript or superscript "0" indicates that the labelled function has been evaluated with zero quark current mass.

Following this procedure one obtains

$$\begin{aligned} C_0 &= 0.121, \quad \overline{m} = 0.00897, \quad C_{\overline{m}} = 0, \\ b_0 &= 0.131, \quad b_1 = 2.90, \quad b_2 = 0.603, \quad b_3 = 0.185. \end{aligned} \quad (21)$$

The mass scale is set by requiring that f_π take its experimental value of 92.4 MeV, which yields $D = 0.160 \text{ GeV}^2$. The calculated values of observables are presented, for completeness, in Table I. The form factor is discussed in Refs. [10,25].

The amplitude in Eq. (1) is given by a four-dimensional integral that we evaluate by straightforward Gaussian quadrature. The calculation is simplified by working in the rest-frame of one of the outgoing pions. All results reported here are tested against variation of numerical parameters, such as grid densities, etc.

The quark 2-point Schwinger function specified by Eqs. (2), (4) and (5) is an entire function. The quantity $x[\bar{\sigma}_V^0(x)]^2 + [\bar{\sigma}_S^0(x)]^2$ has a pair of complex conjugate zeros at $x = -0.102 \pm i 0.715$, which corresponds to $|p^2 = x/(2D)| \simeq 12 m_\pi^2$. This affects our calculation through the approximation of Eq. (10) because $B_0(p^2)$ has poles at these points. These poles are integrable singularities and do not lead to an imaginary part in the amplitude. However, in our calculation of $F^{3\pi}(s, t, u)$ we treat them as a spurious artifact of the approximation of Eq. (10) and consider our results to be unreliable when these poles enter the integration region. This occurs only for $\bar{s} > 12$. The approximation of Eq. (10) has hitherto only been tested for real, spacelike- p^2 and the present application involves a considerable extrapolation into the complex- p^2 plane as \bar{s} is increased. (In the chiral limit, $m_\pi^2 = 0$, the integration does not explore the complex- p^2 plane and these poles do not enter the integration region.) The importance of this observation is that processes such as $\gamma\pi^* \rightarrow \pi\pi$ provide a means of exploring the structure of bound state amplitudes in the complex- p^2 plane.

Our numerical results are shown in Fig. 2. The solid line is our calculated result for the $\gamma\pi^*\pi\pi$ amplitude $\tilde{F}^{3\pi}(s, t, u)$, Eq. (16), as function of the Mandelstam variable \bar{s} for different values of \bar{u} as given by Eq. (18) at $\bar{t} = -1$. An excellent fit to this curve is given by

$$\tilde{F}^{3\pi}(s, t, u) = 1.044 + 0.096 x + 0.006 x^2 \quad \text{with} \quad x = (\bar{s}/4 - 1). \quad (22)$$

Fixing the Mandelstam variables such that the energy of the incident photon is 2 GeV in the proton rest-frame changes our curve by an amount that is not visible on the scale of this plot. Importantly, the result is *insensitive* to the details of the parametrisation of the quark 2-point Schwinger function. A quantitatively similar curve is obtained using earlier sets of the parameters in Eq. (21); i.e., our result is not sensitive to the details of the model. The form factor does depend on the pion mass. The short-dashed line is the result we obtain with $m_\pi = m_\pi^{\text{exp}}/2$.

A comparison of our result with that obtained in other models reveals that all results are broadly consistent. Our prediction for this form factor is, however, uniformly larger and displays a more rapid increase with \bar{s} . For example, using vector meson dominance one obtains, for real photons [26],

$$\tilde{F}^{3\pi}(s, t, u) = 1 + C_\rho e^{i\delta} \left(\frac{s}{m_\rho^2 - s} + \frac{t'}{m_\rho^2 - t'} + \frac{u}{m_\rho^2 - u} \right) \quad (23)$$

with $t' = s - t + u - 2m_\pi^2$, $C_\rho = 2g_{\rho\pi\pi}g_{\rho\pi\gamma}/[m_\rho^3 F^{3\pi}(0, 0, 0)] \approx 0.434$ and δ an unknown phase. This expression, with $\delta = 0$, appears as the long-dashed line in Fig. 2. This curve is smaller in magnitude than our result for all \bar{s} and rises more slowly with \bar{s} . Another vector meson dominance model estimate of this process [27] appears as the dash-dot line in Fig. 2. This result is obtained from

$$\tilde{F}^{3\pi}(s, t, u) = \frac{m_\rho^2}{3} \left(\frac{1}{m_\rho^2 - s} + \frac{1}{m_\rho^2 - t'} + \frac{1}{m_\rho^2 - u} \right). \quad (24)$$

and is even smaller in magnitude and has a weaker \bar{s} dependence than the other vector meson dominance estimate.

The weakest \bar{s} dependence is obtained using a model based on chiral expansion techniques and employing a vector meson saturation Ansatz [28]. This result appears as the dotted line in Fig. 2 and is obtained from the expression

$$\tilde{F}^{3\pi}(s, t, u) = \left| 1 + C_{\text{pion-loops}} + \frac{s + t' + u}{2m_\rho^2} \right| \quad (25)$$

where the non-divergent part of the coefficient $C_{\text{pion-loops}}$ is

$$C_{\text{pion-loops}} = \frac{1}{96\pi^2 f_\pi^2} \left[\left(\log \left[\frac{m_\rho^2}{m_\pi^2} \right] + \frac{5}{3} \right) (s + t' + u) + 4m_\pi^2 \{f(s) + f(t') + f(u)\} \right] \quad (26)$$

with $f(\zeta < 0)$ defined by

$$f(\zeta) = (1 - \zeta/4m_\pi^2) \sqrt{1 - 4m_\pi^2/\zeta} \log \left(\frac{\sqrt{1 - 4m_\pi^2/\zeta} + 1}{\sqrt{1 - 4m_\pi^2/\zeta} - 1} \right) - 2. \quad (27)$$

Analytic continuation is used to define $f(\zeta > 0)$. Note that $f(\zeta)$ has an imaginary part for $\zeta > 4m_\pi^2$; i.e., for s in the domain explored by the existing and proposed experiments. The imaginary part is due to the pion loop in the s -channel.

We can compare our result with the one existing data point [5]. Its statistical and systematic error as well as the uncertainty in s is also displayed in Fig. 2. The fact that this data point is well above the chiral limit prediction has caused some concern [7]. However, given the experimental errors and the prediction of our model this data point does not appear untenable.

5. Summary and Conclusions. Herein we have reported a calculation of the form factor for the anomalous process $\gamma\pi^* \rightarrow \pi\pi$, $F^{3\pi}(s, t, u)$, in a phenomenological approach based on the QCD Dyson-Schwinger equations (DSEs). In our approach the chiral limit value of $F^{3\pi}(0, 0, 0)$ is reproduced *independent* of the details of the quark 2-point Schwinger function. Using for the u - d -quark 2-point Schwinger function a parametrisation fitted to low-energy pion data, $F^{3\pi}(s, t, u)$ was calculated for a range of (s, t, u) specified in such a way as to cover the kinematic region to be explored in a proposed experiment [7]. The small photon virtuality in another proposed experiment [8] should also make it possible for our prediction to be compared, without adjustment, with the results of that experiment.

We compared our result with that obtained in other models when applied in the same (s, t, u) range as we have considered, which is the range appropriate for the experiment

proposed at CEBAF [7]. Our model yields a form factor, $F^{3\pi}(s, t, u)$, that is uniformly larger and has a more rapid increase with s than any of the other models considered. The result obtained in a model based on chiral expansion techniques, augmented by vector meson saturation assumptions, leads to the weakest s dependence. Although the models are broadly consistent, the variation between the results is such that the proposed experiments should be able to distinguish between them.

Along with the study in Ref. [3], this phenomenological application of the QCD DSEs is one of the first to explore the model quark 2-point Schwinger function and pion Bethe-Salpeter amplitude in the timelike region, which is not accessible in perturbation theory. This region is important in the study of, for example, vector meson Bethe-Salpeter and baryon Fadde'ev amplitudes. Experimental data on $F^{3\pi}(s, t, u)$ can therefore be used to place additional constraints on the analytic structure of the QCD Schwinger functions.

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TABLES

	Calculated	Experiment
f_π	0.0924 GeV	0.0924 ± 0.001
m_π	0.1385	0.1385
$m_{1\text{GeV}^2}^{\text{ave}}$	0.0051	0.0075
$-\langle\bar{q}q\rangle_{1\text{GeV}^2}^{\frac{1}{3}}$	0.221	0.220
r_π	0.55 fm	0.663 ± 0.006
$g_{\pi^0\gamma\gamma}$	0.505 (dimensionless)	0.504 ± 0.019
$F^{3\pi}(4m_\pi^2)$	1.04	1 (Anomaly)
a_0^0	0.17	0.21 ± 0.01
a_0^2	-0.048	-0.040 ± 0.003
a_1^1	0.030	0.038 ± 0.003
a_2^0	0.0015	0.0017 ± 0.0003
a_2^2	-0.00021	

TABLE I. Pion observables calculated using the parameter values in Eq. (21). The “experimental” values listed for m^{ave} and $\langle\bar{q}q\rangle$ are an indication of other contemporary theoretical estimates. Experimental values not discussed in the text are taken from Ref. [23]. The difference between the calculated and experimental values of r_π is a measure of the importance of final-state π - π interactions and photon- ρ -meson mixing [24]; that between the calculated and experimental values of the pion scattering lengths is a measure of the importance of π - π final-state interactions in this case [20].

FIGURES

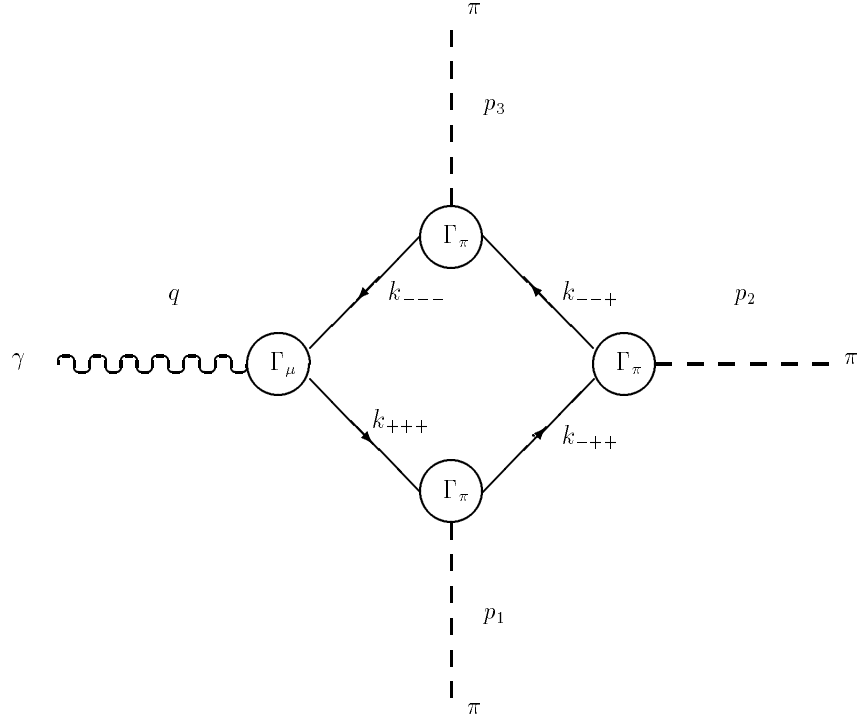


FIG. 1. This figure is a pictorial representation of the amplitude identified with the $\gamma\pi^*\pi\pi$ vertex. The straight broken external lines represent the outgoing π , the circles at the π legs represent the $\langle\pi|\bar{q}q\rangle$ Bethe-Salpeter amplitudes, Γ_π , the wiggly line represents the photon, γ , the circle at the γ leg represents the regular part of the dressed quark-photon vertex, Γ_μ , and the full internal lines represent the dressed quark 2-point Schwinger function, S .

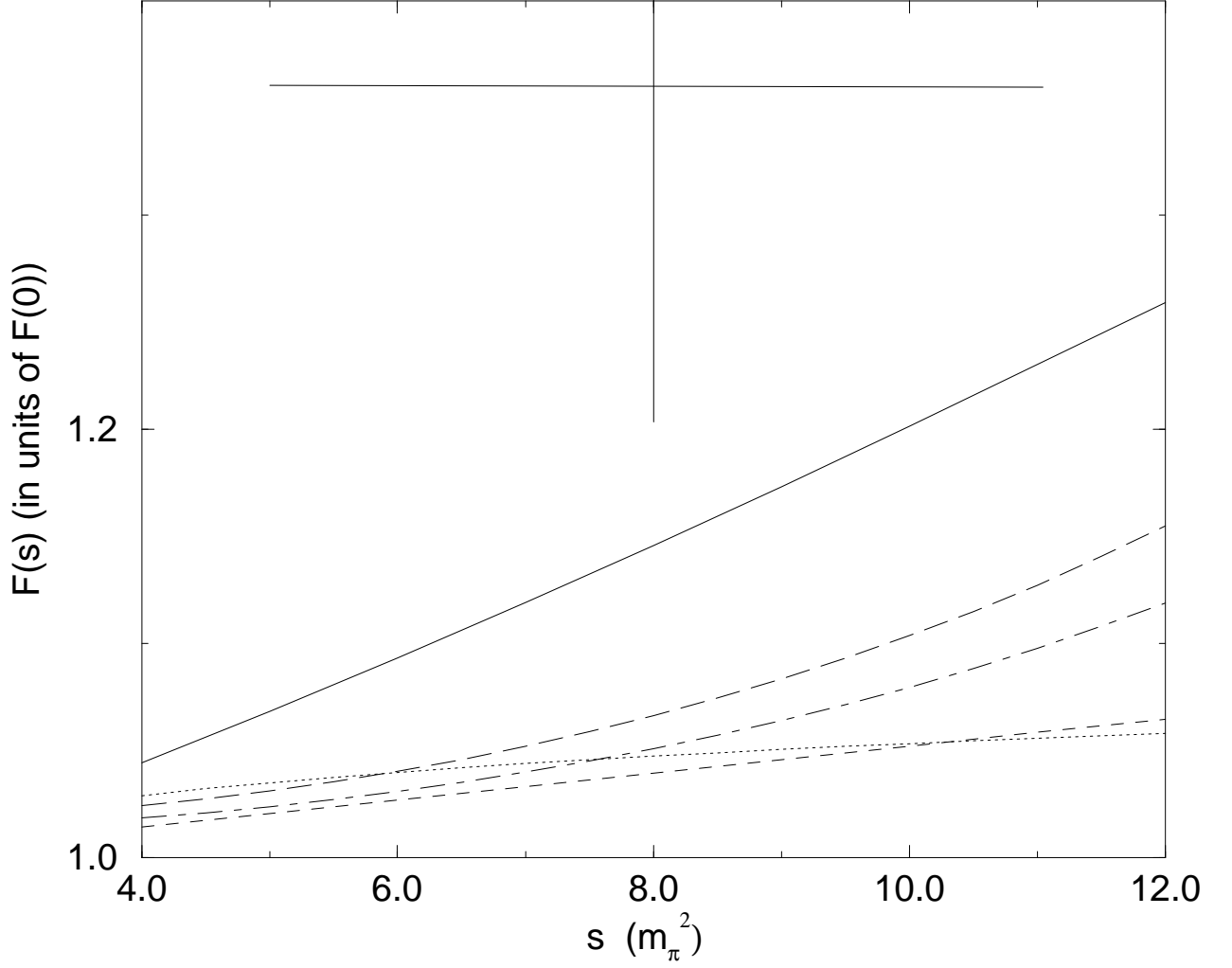


FIG. 2. The normalized $\gamma\pi^*\pi\pi$ amplitude $\tilde{F}^{3\pi}(s, t, u)$, Eq. (16), as function of the Mandelstam variable \bar{s} for different values of \bar{u} as given by Eq. (18) at $\bar{t} = -1$. Our result: solid line; our result with $m_\pi \rightarrow m_\pi/2$: short-dash line; vector meson dominance: long-dash and dash-dot lines, see Eqs. (23) and (24); chiral expansion plus vector meson saturation Ansatz: dotted line, see Eq. (25). The data point is taken from Ref. [5].